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De Sitter thermodynamics from diamonds's temperature

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Abstract: The thermal time hypothesis proposed by Rovelli [1] regards the physical basis for the flow of time as thermodynamical and provides a definition of the temperature for some special cases. We verify this hypothesis in the case of de Sitter spacetime by relating the uniformly accelerated observer in de Sitter spacetime to the diamond in Minkowski spacetime. Then, as an application of it, we investigate the thermal effect for the uniformly accelerated observer with a finite lifetime in dS spacetime, which generalizes the corresponding result for the case of Minkowski spacetime [2].
Furthermore, noticing that a uniformly accelerated dS observer with a finite lifetime corresponds to a Rindler observer with a finite lifetime in the embedding Minkowski spacetime, we show that the global-embedding-Minkowski-spacetime (GEMS) picture of spacetime thermodynamics is valid in this case. This is a rather nontrivial and unexpected generalization of the GEMS picture, as well as a further verification of both the thermal time hypothesis and the GEMS picture.

Keywords: Classical Theories of Gravity, Models of Quantum Gravity.

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Contents

1. Introduction

The thermodynamics of spacetimes has drawn much interest after the discovery of Hawking radiation [3] and Unruh effect [4]. The thermodynamics of de Sitter (dS) spacetime [5] is especially amazing and leads to a lot of puzzles [6], which are made more pressing by recent cosmological observations showing that our universe is probably asymptotically dS [7].

Recently, Martinetti and Rovelli reviewed the Unruh effect from the viewpoint of the so-called "thermal time hypothesis" [1] and applied this hypothesis to study the Unruh effect for an observer with a finite lifetime, which they call the "diamonds's temperature" [2]. In this paper, we explore the possibility of extending the application of this hypothesis to dS spacetime. First, we find that it gives the correct (constant) temperature for a uniformly accelerated observer in dS spacetime, as the evidence of its validity in this case. Then, we go on to apply it to a uniformly accelerated observer with a finite lifetime in dS spacetime. Similar to the result in [2], we explicitly obtain a time-dependent temperature for this kind of observers.

Unexpectedly, we find an elegant relation between our result and the so-called global-embedding-Minkowski-spacetime (GEMS) picture of spacetime thermodynamics [8, 9]. First, the temperature for the uniformly accelerated dS observer is consistent with the known result [8] obtained from the GEMS picture. Next, noticing that a uniformly accelerated dS observer with a finite lifetime corresponds to a Rindler observer with a finite lifetime in the embedding Minkowski spacetime, we show that the GEMS picture is valid in this case. In other words, the thermal time hypothesis and the GEMS picture are compatible with each other, at least in this case. Although both their physical meanings are not very clear, this compatibility seems to justify both of them to some extent.

2. Thermodynamics of an eternal observer in dS spacetime from diamonds's temperature

The key observation is that the region covered by the static coordinates, with metric

$$ds^2 = \left(1 - \frac{r^2}{R^2}\right) dt^2 - \frac{1}{1 - \frac{r^2}{R^2}} dr^2 - r^2 d\Omega^2 \,, \tag{2.1}$$

on dS spacetime can be mapped to a diamond (which we call the dS diamond) on Minkowski spacetime via a conformal transformation which maps the dS spacetime to Minkowski spacetime in the sense of conformal compactification.[1] From the 5-dimensional viewpoint that regards the dS spacetime as a pseudo-sphere

$$\eta_{AB}\xi^A\xi^B = -R^2 \,, \tag{2.2}$$

$$ds^2 = \eta_{AB} d\xi^A d\xi^B \tag{2.3}$$

with $\eta_{AB} = \mathrm{diag}(1, -1, -1, -1, -1)$, this conformal transformation is simply realized by a (pseudo-)stereographic projection,

$$x^\mu = \frac{2R\xi^\mu}{R + \xi^4} \,, \qquad \mu = 0, 1, \ldots, 3 \,, \tag{2.4}$$

from the point $P = (0, 0, 0, 0, -R)$ to the hyperplane $\xi^4 = R$ with Minkowski signature, where x^μ stand for the coordinates ξ^μ on this hyperplane. At the same time, x^μ can be taken as coordinates on the dS spacetime, known as the conformally flat coordinates. This coordinate system can cover almost the entire dS spacetime, with metric

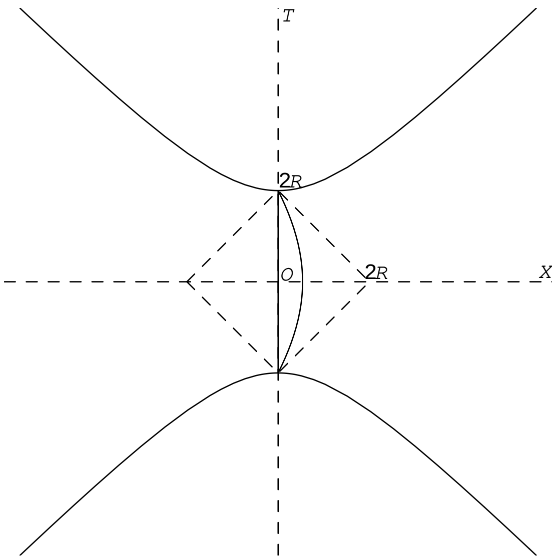
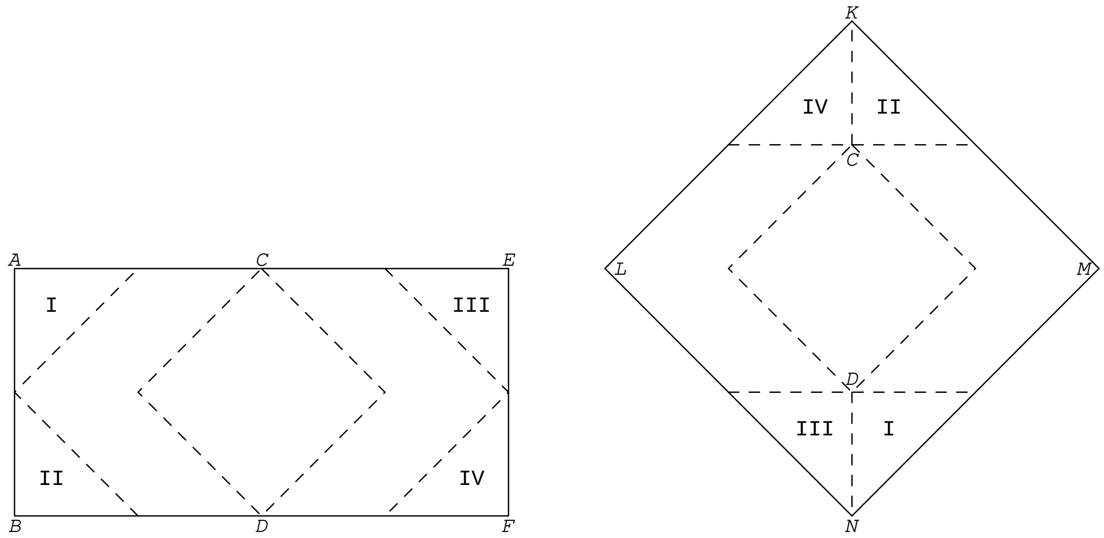


Figure 1: A two-dimensional sketch map of the conformally flat coordinates on the dS spacetime, where we have taken the notation $T \equiv x^0$ and $X \equiv x^1$. All the points on the plane except those on the hyperbola with equation $c(x) = 1 - \frac{T^2 - X^2}{4R^2} = 0$, which is actually the conformal boundary of the dS spacetime, are points on the dS spacetime. The diamond embraced by the dashed lines is the region covered by the static coordinates. The solid line segment is the world line of the inertial observer, while the solid segment of a hyperbola is the world line of the observer staying at $r = R/2$. When $r \to R$, the world line (2.7) of the uniformly accelerated observer tends to the boundary of the diamond, $T^2 = (X - 2R)^2$.

$$ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{c^2(x)} \,, \qquad c(x) \equiv 1 - \frac{\eta_{\mu\nu} x^\mu x^\nu}{4R^2} \,, \tag{2.5}$$

except the light cone at P. For a sketch map of the conformally flat coordinates, see figure 1. In terms of Penrose diagrams, one can simply illustrate the relation between the conformal compactificaion of the dS spacetime and that of the Minkowski spacetime induced by the stereographic projection (see figure 2).

[1] Concerning the role played by conformal transformations, [2] restricts itself to conformally invariant quantum field theories. Since the spacetime thermodynamics is mainly determined by the causal structure and conformal transformations preserve the latter, in fact, one may argue that the restriction can be removed.



(a) Penrose diagram of the dS spacetime, with identification $AB = EF$, and with ACE and BDF its conformal boundary. It is conformally compactified by identifying $AC = DF$ and $BD = CE$. The small diamond with vertices C and D is the region covered by the static coordinates.

(b) Penrose diagram of the Minkowski spacetime, which can be compared with fig. 1, with the horizontal dashed lines here corresponding to the conformal boundary of the dS spacetime (the hyperbola) there. It is conformally compactified by identifying $KL = MN$ and $KM = LN$. What the stereographic projection does is just to remove the triangles (numbered "I" to "IV") in fig. 2a to the corresponding positions in this figure, respectively.

Figure 2: An illustration of the stereographic projection in terms of Penrose diagrams.

As is well known, the observer with world line $r = \text{const}$ in the static coordinates is a uniformly accelerated observer in the dS spacetime, with constant acceleration

$$a = \frac{r}{R(R^2 - r^2)^{1/2}} . \tag{2.6}$$

Especially, the observer staying at $r = 0$ is of acceleration $a = 0$, i.e., an inertial observer, and the acceleration diverges when $r \to R$. In terms of the conformally flat coordinates, the equation of the world line $r = \text{const}$ is

$$T^2 = X^2 - \frac{4R^2}{r} X + 4R^2 , \tag{2.7}$$

where we have restricted ourselves to the two-dimensional case for simplicity. Taking r as a free parameter, the above equation is the set of all pseudo-circles[2] that passing the points

[2]By the term pseudo-circle, in fact, we mean a hyperbola with equation

$$(T - T_0)^2 - (X - X_0)^2 = W,$$

$(0, -2R)$ and $(0, 2R)$. See figure 1 for the world lines of uniformly accelerated observers in the conformally flat coordinates.

Now it is easy to obtain the action of the modular group (see eq. (45) of [2]) on the dS diamond:

$$T(\rho) = \frac{2R\sinh\rho}{\cosh\rho + R(R^2 - r^2)^{-1/2}}, \tag{2.8}$$

$$X(\rho) = \frac{2Rr(R^2 - r^2)^{-1/2}}{\cosh\rho + R(R^2 - r^2)^{-1/2}}. \tag{2.9}$$

One can check that this modular flow is along the world line (2.7), which is the precondition of a well-defined temperature from the thermal time hypothesis. On the other hand, we have from eq. (2.5)

$$ds^2 = (dT^2 - dX^2)\left(1 - \frac{T^2 - X^2}{4R^2}\right)^{-2}, \tag{2.10}$$

which leads to

$$\frac{ds}{dT} = \frac{2R(R^2 - r^2)^{1/2}[2R^2 + (4R^4 - 4R^2r^2 + r^2T^2)^{1/2}]}{(4R^2 - T^2)(4R^4 - 4R^2r^2 + r^2T^2)^{1/2}}. \tag{2.11}$$

In order to apply the thermal time hypothesis [2], we need to obtain $d\rho/dT$, in terms of T, from eq. (2.8).[3] After some simplification, the final result is

$$\frac{d\rho}{dT} = \frac{2R[2R^2 + (4R^4 - 4R^2r^2 + r^2T^2)^{1/2}]}{(4R^2 - T^2)(4R^4 - 4R^2r^2 + r^2T^2)^{1/2}}. \tag{2.12}$$

The thermal time hypothesis then gives

$$\beta = 2\pi\frac{ds}{d\rho} = 2\pi(R^2 - r^2)^{1/2}, \tag{2.13}$$

which is independent of T (constant for the observer) and in exact agreement with the known result [8].

3. Thermodynamics of an observer with finite lifetime in dS spacetime from diamonds's temperature

The above agreement, which at least shows some evidence of the validity of the thermal time hypothesis in the dS spacetime, stimulates us to go further along this direction, i.e., to consider a uniformly accelerated observer with finite lifetime in the dS spacetime, in order to generalize the result in [2]. It can be shown that the static observer with lifetime $-\tau \leq t \leq \tau$ in the static coordinates is associated with a reduced diamond with half height

$$2M = \frac{2R\sinh(\tau/R)}{\cosh(\tau/R) + R(R^2 - r^2)^{-1/2}}, \tag{3.1}$$

where T_0, X_0 and W are arbitrary real constants.

[3]It seems elementary to fulfill this task, but one will encounter considerable difficulty if not appealing to some algebraic tricks.

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which is smaller than $2R$ and tends to $2R$ when $\tau \to \infty$. Now the modular flow corresponding to the reduced diamond is

$$T(\rho) = \frac{2M \sinh \rho}{\cosh \rho + \dfrac{R(R^2 - r^2)^{-1/2} \cosh(\tau/R) + 1}{\cosh(\tau/R) + R(R^2 - r^2)^{-1/2}}}\,, \tag{3.2}$$

instead of eq. (2.8). Thus we can obtain $d\rho/dT$, by making the replacement[4]

$$R \to M\,, \qquad R(R^2 - r^2)^{-1/2} \to \frac{R(R^2 - r^2)^{-1/2} \cosh(\tau/R) + 1}{\cosh(\tau/R) + R(R^2 - r^2)^{-1/2}}\,, \tag{3.3}$$

as

$$\frac{d\rho}{dT} = \frac{2R\left\{2R^2\left[\cosh\frac{\tau}{R} + \frac{(R^2-r^2)^{1/2}}{R}\right] + F\left[\cosh\frac{\tau}{R} + \frac{R}{(R^2-r^2)^{1/2}}\right]\right\}\sinh\frac{\tau}{R}}{\left\{4R^2\sinh^2\left(\frac{\tau}{R}\right) - T^2\left[\cosh\left(\frac{\tau}{R}\right) + R(R^2-r^2)^{-1/2}\right]^2\right\}F}\,, \tag{3.4}$$

where we have defined

$$F \equiv (4R^4 - 4R^2r^2 + r^2T^2)^{1/2}\,. \tag{3.5}$$

Then from eq. (2.11) and the thermal time hypothesis we have

$$\beta = \frac{2\pi(R^2-r^2)^{1/2}(2R^2+F)\left\{4R^2\sinh^2\frac{\tau}{R} - T^2\left[\cosh\frac{\tau}{R} + \frac{R}{(R^2-r^2)^{1/2}}\right]^2\right\}}{\left\{2R^2\left[\cosh\frac{\tau}{R} + \frac{(R^2-r^2)^{1/2}}{R}\right] + F\left[\cosh\frac{\tau}{R} + \frac{R}{(R^2-r^2)^{1/2}}\right]\right\}(4R^2 - T^2)\sinh\frac{\tau}{R}}\,. \tag{3.6}$$

This formula looks complicated, but we will see in the next section that it can be thoroughly simplified in terms of the static time t, with an elegant relation to the so-called GEMS picture. In fact, the temperature has similar time dependence as that in [2], i.e., diverges at the two ends of the lifetime ($T = \pm 2M$) and has a minimum at $T = 0$ with

$$\beta = 2\pi(R^2 - r^2)^{1/2}\tanh\frac{\tau}{2R}\,. \tag{3.7}$$

As a primary check of the complicated expression (3.6), we take its limit of $R \to \infty$ with r fixed, and then get

$$\beta = \pi\frac{\tau^2 - T^2}{\tau}\,, \tag{3.8}$$

which is actually the same as the $a \to 0$ limit of eq. (54) in [2]. Its $a \neq 0$ case is not so straightforward, since in this case r tends to infinity as R does, and then T does not tend to the proper time. First we have

$$s = (1 - r^2/R^2)^{1/2}t = (1 + a^2R^2)^{-1/2}t \tag{3.9}$$

[4]It implies

$$r \to \frac{r\sinh^2(\tau/R)}{[\cosh(\tau/R) + (R^2-r^2)^{1/2}/R][\cosh(\tau/R) + R(R^2-r^2)^{-1/2}]}\,.$$

with s the proper time, and so

$$T = \frac{2R\sinh(t/R)}{\cosh(t/R) + R(R^2 - r^2)^{-1/2}} \tag{3.10}$$

$$\rightarrow \frac{2\sinh(as)}{a}\,. \tag{3.11}$$

Thus follows

$$F \rightarrow [4R^2a^{-2} + 4R^2a^{-2}\sinh^2(as)]^{1/2} = 2Ra^{-1}\cosh(as)\,. \tag{3.12}$$

Substituting the above equations into eq. (3.6) and taking the $R \rightarrow \infty$ limit, we have finally

$$\beta = 2\pi\frac{\cosh(a\sigma) - \cosh(as)}{a\sinh(a\sigma)} \tag{3.13}$$

with the proper time range

$$-\sigma \leq s \leq \sigma\,, \tag{3.14}$$

which is actually the same as eq. (54) in [2]. That shows eq. (3.6) has the correct flat limit.

Although the above discussion assumes the spacetime dimension to be four, it does not depend on that, so it can be easily extended to any spacetime dimensions.

4. Embedding of the dS observers in Minkowski spacetime

The global-embedding-Minkowski-spacetime (GEMS) picture of spacetime thermodynamics has been extensively studied in the literature [8−11], since it was first confirmed in the case of dS spacetime [12, 13], due to the key observation that an inertial observer in a d-dimensional dS spacetime corresponds to a Rindler observer in the $(d+1)$-dimensional embedding Minkowski spacetime. This picture matches the Hawking temperature detected by a stationary observer in a curved spacetime with the Unruh temperature detected by the corresponding observer in the GEMS of that spacetime.

Since it is obvious that a uniformly accelerated observer with a finite lifetime in the d-dimensional dS spacetime corresponds to a Rindler observer with a finite lifetime in the $(d+1)$-dimensional GEMS and both their thermal effects have been known, it is natural and interesting to ask whether these two thermal effects can match each other.

For a Rindler observer with the finite (proper) lifetime (3.14) in the 5-dimensional Minkowski spacetime, we have from eq. (3.13)

$$\beta = 2\pi\frac{\cosh(a_5\sigma) - \cosh(a_5 s)}{a_5\sinh(a_5\sigma)}\,. \tag{4.1}$$

On the other hand, for the uniformly accelerated observer staying at $r = \text{const}$ in the static dS spacetime, we have for its GEMS [8]

$$a_5 = (a^2 + R^{-2})^{1/2} = (R^2 - r^2)^{-1/2}\,. \tag{4.2}$$

Recalling eq. (3.9) then reduces eq. (4.1) to

$$\beta = 2\pi(R^2 - r^2)^{1/2}\frac{\cosh(\tau/R) - \cosh(t/R)}{\sinh(\tau/R)} \tag{4.3}$$

with the static time range $-\tau \le t \le \tau$. This expression is much simpler than eq. (3.6), but they are actually the same when taking into account the relation (3.10) between the conformally flat time T and the static one t, which may be a little surprising. We leave the detailed computation in the appendix. Note that the identification of eq. (3.6) and eq. (4.3) is nontrivial, since we have obtained them from different approaches: the former from the stereographic projection and the latter from the GEMS picture. (Of course, both of them depend on the thermal time hypothesis.)

It is striking that the GEMS picture of spacetime thermodynamics is valid for a uniformly accelerated observer with a finite lifetime in the dS spacetime. The GEMS picture has only been confirmed in many static cases for a long time, until [10] points out that it can be valid for general stationary motions with not only the temperatures but also the chemical potentials matched. At the same time, there is strong evidence that the GEMS picture fails for non-stationary motions [11]. Now the uniformly accelerated dS observer with a finite lifetime serves as an example that the GEMS picture can be generalized in another way.

5. Concluding remarks

The thermodynamics of dS spacetime itself (i.e., not with respect to a specific observer) can be investigated from various viewpoints: the event horizon [5], the GEMS of (eternal) static observers [8], and the thermal time hypothesis for (eternal) static observers as in section 2. All these viewpoints lead to the same result (2.13), which strongly justifies the puzzling dS thermodynamics. At the same time, this can be regarded as a verification of the thermal time hypothesis in a non-flat spacetime, the significance of which should be explored further.

Noticing that a uniformly accelerated dS observer with a finite lifetime corresponds to a Rindler observer with a finite lifetime in the GEMS, we have shown that the GEMS picture of spacetime thermodynamics is valid in this case, which is a whole new result of this paper. On one hand, this result can be regarded as a further (and rather unexpected) verification of both the thermal time hypothesis and the GEMS picture. On the other hand, it is very interesting to study whether the GEMS picture can be generalized any further.

Acknowledgments

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A. Simplification of eq. (3.6) in terms of the static time t

First, substitution of the relation (3.10) between the conformally flat time T and the static one t into eq. (3.6) gives

$$\begin{aligned}\beta = & \frac{2\pi(R^2-r^2)^{1/2}(2R^2+F)}{\left\{2R^2\left[\cosh\frac{\tau}{R}+\frac{(R^2-r^2)^{1/2}}{R}\right]+F\left[\cosh\frac{\tau}{R}+\frac{R}{(R^2-r^2)^{1/2}}\right]\right\}}\times \\ & \times\frac{\left\{\sinh^2\frac{\tau}{R}\left[\cosh\frac{t}{R}+\frac{R}{(R^2-r^2)^{1/2}}\right]^2-\sinh^2\frac{t}{R}\left[\cosh\frac{\tau}{R}+\frac{R}{(R^2-r^2)^{1/2}}\right]^2\right\}}{\left\{\left[\cosh\frac{t}{R}+\frac{R}{(R^2-r^2)^{1/2}}\right]^2-\sinh^2\frac{t}{R}\right\}\sinh\frac{\tau}{R}}.\end{aligned} \tag{A.1}$$

Then, substituting eq. (3.10) into the definition (3.5) of F in terms of T, we have

$$\begin{aligned}F &= \sqrt{4R^2(R^2-r^2)+4R^2r^2\frac{\sinh^2\left(\frac{t}{R}\right)}{\left[\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}\right]^2}} \\ &= 2R\frac{\sqrt{(R^2-r^2)\left[\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}\right]^2+r^2\sinh^2\left(\frac{t}{R}\right)}}{\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}} \\ &= 2R\frac{\sqrt{R^2\cosh^2\left(\frac{t}{R}\right)+2R(R^2-r^2)^{1/2}\cosh\left(\frac{t}{R}\right)+R^2-r^2}}{\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}} \\ &= 2R^2\frac{\cosh\left(\frac{t}{R}\right)+(R^2-r^2)^{1/2}/R}{\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}},\end{aligned} \tag{A.2}$$

which leads to

$$2R^2+F = 2R^2\frac{\left[\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}\right]+\left[\cosh\left(\frac{t}{R}\right)+\frac{(R^2-r^2)^{1/2}}{R}\right]}{\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}} \tag{A.3}$$

and

$$\begin{aligned}&2R^2\left[\cosh\left(\frac{\tau}{R}\right)+\frac{(R^2-r^2)^{1/2}}{R}\right]+F\left[\cosh\left(\frac{\tau}{R}\right)+R(R^2-r^2)^{-1/2}\right]= \\ &\qquad = 2R^2\left(\frac{\left[\cosh\frac{\tau}{R}+\frac{(R^2-r^2)^{1/2}}{R}\right]\left[\cosh\frac{t}{R}+\frac{R}{(R^2-r^2)^{1/2}}\right]}{\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}}+\right. \\ &\qquad\qquad \left.+\frac{\left[\cosh\frac{t}{R}+\frac{(R^2-r^2)^{1/2}}{R}\right]\left[\cosh\frac{\tau}{R}+\frac{R}{(R^2-r^2)^{1/2}}\right]}{\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}}\right).\end{aligned}$$

On the other hand, we have

$$\left[\cosh\left(\frac{t}{R}\right)+R(R^2-r^2)^{-1/2}\right]^2-\sinh^2\left(\frac{t}{R}\right)=$$

$$
\begin{aligned}
&= 1 + 2R(R^2 - r^2)^{-1/2} \cosh\left(\frac{t}{R}\right) + \frac{R^2}{(R^2 - r^2)} \\
&= R(R^2 - r^2)^{-1/2} \left\{ \left[\cosh\left(\frac{t}{R}\right) + R(R^2 - r^2)^{-1/2}\right] + \left[\cosh\left(\frac{t}{R}\right) + \frac{(R^2 - r^2)^{1/2}}{R}\right] \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\sinh^2\left(\frac{\tau}{R}\right) \left[\cosh\left(\frac{t}{R}\right) + R(R^2 - r^2)^{-1/2}\right]^2 - \sinh^2\left(\frac{t}{R}\right) \left[\cosh\left(\frac{\tau}{R}\right) + R(R^2 - r^2)^{-1/2}\right]^2 = \\
= \left[\cosh^2\left(\frac{\tau}{R}\right) - 1\right] \left[\cosh\left(\frac{t}{R}\right) + R(R^2 - r^2)^{-1/2}\right]^2 - \\
- \left[\cosh^2\left(\frac{t}{R}\right) - 1\right] \left[\cosh\left(\frac{\tau}{R}\right) + R(R^2 - r^2)^{-1/2}\right]^2 \\
= 2\cosh^2\left(\frac{\tau}{R}\right) \cosh\left(\frac{t}{R}\right) R(R^2 - r^2)^{-1/2} + \cosh^2\left(\frac{\tau}{R}\right) \frac{R^2}{(R^2 - r^2)} - \\
- \cosh^2\left(\frac{t}{R}\right) - 2\cosh\left(\frac{t}{R}\right) R(R^2 - r^2)^{-1/2} - \frac{R^2}{(R^2 - r^2)} - \\
- 2\cosh^2\left(\frac{t}{R}\right) \cosh\left(\frac{\tau}{R}\right) R(R^2 - r^2)^{-1/2} - \cosh^2\left(\frac{t}{R}\right) \frac{R^2}{(R^2 - r^2)} + \\
+ \cosh^2\left(\frac{\tau}{R}\right) + 2\cosh\left(\frac{\tau}{R}\right) R(R^2 - r^2)^{-1/2} + \frac{R^2}{(R^2 - r^2)} \\
= R(R^2 - r^2)^{-1/2} \left[2\cosh\left(\frac{\tau}{R}\right) \cosh\left(\frac{t}{R}\right) + \cosh\left(\frac{\tau}{R}\right) \frac{(R^2 - r^2)^{1/2}}{R} + \right. \\
+ \cosh\left(\frac{t}{R}\right) \frac{(R^2 - r^2)^{1/2}}{R} + \\
+ \cosh\left(\frac{\tau}{R}\right) R(R^2 - r^2)^{-1/2} + \\
\left. + \cosh\left(\frac{t}{R}\right) R(R^2 - r^2)^{-1/2} + 2 \right] \times \\
\times \left[\cosh\left(\frac{\tau}{R}\right) - \cosh\left(\frac{t}{R}\right)\right] \\
= R(R^2 - r^2)^{-1/2} \left\{ \left[\cosh\left(\frac{\tau}{R}\right) + \frac{(R^2 - r^2)^{1/2}}{R}\right] \times \right. \\
\times \left[\cosh\left(\frac{t}{R}\right) + R(R^2 - r^2)^{-1/2}\right] + \\
+ \left[\cosh\left(\frac{t}{R}\right) + \frac{(R^2 - r^2)^{1/2}}{R}\right] \times \\
\left. \times \left[\cosh\left(\frac{\tau}{R}\right) + R(R^2 - r^2)^{-1/2}\right] \right\} \times \\
\times \left[\cosh\left(\frac{\tau}{R}\right) - \cosh\left(\frac{t}{R}\right)\right] .
\end{aligned}
$$

Finally, substitution of the above four expressions into eq. (A.1) gives

$$\beta = 2\pi(R^2 - r^2)^{1/2}\frac{\cosh\left(\frac{\tau}{R}\right) - \cosh\left(\frac{t}{R}\right)}{\sinh(\tau/R)}\,, \tag{A.4}$$

where the other complicated factors have all reduced out. This thoroughly simplified formula is exactly the same as eq. (4.3).

References

[1] C. Rovelli, *Statistical mechanics of gravity and the thermodynamical origin of time*, *Class. and Quant. Grav.* **10** (1993) 1549; *The statistical state of the universe*, *Class. and Quant. Grav.* **10** (1993) 1567.

[2] P. Martinetti and C. Rovelli, *Diamonds's temperature: Unruh effect for bounded trajectories and thermal time hypothesis*, *Class. and Quant. Grav.* **20** (2003) 4919 [gr-qc/0212074].

[3] S.W. Hawking, *Particle creation by black holes*, *Commun. Math. Phys.* **43** (1975) 199.

[4] W.G. Unruh, *Notes on black hole evaporation*, *Phys. Rev.* **D 14** (1976) 870.

[5] G.W. Gibbons and S.W. Hawking, *Cosmological event horizons, thermodynamics and particle creation*, *Phys. Rev.* **D 15** (1977) 2738.

[6] E. Witten, *Quantum gravity in de Sitter space*, hep-th/0106109;
A. Strominger, *Talk at the string satellite*, Conference to ICM 2002, Beijing 2002;
T. Jacobson and R. Parentani, *Horizon entropy*, *Found. Phys.* **33** (2003) 323–348 [gr-qc/0302099];
H.-Y. Guo, C.-G. Huang and B. Zhou, *On horizon in de Sitter invariant special relativity*, hep-th/0404010.

[7] Supernova Cosmology Project collaboration, S. Perlmutter et al., *Measurements of omega and lambda from* 42 *high-redshift supernovae*, *Astrophys. J.* **517** (1999) 565 [astro-ph/9812133];
C.L. Bennett et al, *First year Wilkinson microwave anisotropy probe (WMAP) observations: preliminary maps and basic results*, *Astrophys. J.* **148** (2003) 1 [astro-ph/0302207];
SDSS collaboration, M. Tegmark et al., *Cosmological parameters from sdss and wmap*, *Phys. Rev.* **D 69** (2004) 103501 [astro-ph/0310723].

[8] S. Deser and O. Levin, *Accelerated detectors and temperature in (anti) de Sitter spaces*, *Class. and Quant. Grav.* **14** (1997) L163 [gr-qc/9706018].

[9] S. Deser and O. Levin, *Equivalence of Hawking and Unruh temperatures through flat space embeddings*, *Class. and Quant. Grav.* **15** (1998) L85 [hep-th/9806223]; *Mapping Hawking into Unruh thermal properties*, *Phys. Rev.* **D 59** (1999) 064004 [hep-th/9809159];
Y.-W. Kim, Y.-J. Park and K.-S. Soh, *Eight dimensional flat embedding of reissner-nordstroem-AdS black hole*, *Phys. Rev.* **D 62** (2000) 104020 [gr-qc/0001045];
S.-T. Hong, W.T. Kim, J.J. Oh and Y.-J. Park, *Higher dimensional flat embeddings of black strings in* $(2+1)$ *dimensions*, *Phys. Rev.* **D 63** (2001) 127502 [hep-th/0103036];
T. Padmanabhan, *Thermodynamics and/of Horizons: a comparison of Schwarzschild,Rindler and de Sitter spacetimes*, *Mod. Phys. Lett.* **A 17** (2002) 923;
N.L. Santos, O.J.C. Dias and J.P.S. Lemos, *Global embedding of D-dimensional black holes with a cosmological constant in minkowskian spacetimes: matching between Hawking temperature and Unruh temperature*, *Phys. Rev.* **D 70** (2004) 124033 [hep-th/0412076].

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[10] H.-Z. Chen, Y. Tian, Y.-H. Gao and X.-C. Song, *The gems approach to stationary motions in the spherically symmetric spacetimes*, *JHEP* **10** (2004) 011 [gr-qc/0409107].

[11] H.-Z. Chen and Y. Tian, *Note on the generalization of the global embedding Minkowski spacetime approach*, *Phys. Rev.* **D 71** (2005) 104008 [gr-qc/0410077].

[12] N. Birrell and P. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge, 1982.

[13] H. Narnhofer, I. Peter and W. Thirring, *How hot is the de Sitter space?*, *Int. J. Mod. Phys.* **B 10** (1996) 1507.